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The combined system consisting of the baseline flexible structure modified by the system of active controllers is considered as a unified dynamical system. Techniques based on computer algebra (MACSYMA) are used to derive expressions for the transfer functions of the modified system, using the known transfer functions of the baseline flexible structure and the feedback gains of the active controller. The roots of the characteristic polynomial of this transfer function give the system resonant frequencies and damping parameters. Using the computer algebraic system MACSYMA, expressions for these parameters which are explicitly dependent on the output feedback gains of the active controller, are presented. For lightly couped modes, simple relations are obtained between the modal parameters and the coordinates of the sensor/actuator pairs as well as the displacement and velocity feedback gains. These results permit the parametric study of the placement of the resonant frequencies and damping parameters of the combined system, as functions of the feedback gains. Numerical examples are used to illustrate the application of these results to the calculation of active controller feedback gains based on the requirement that certing modes have specified modal damping while the closed-loop frequencies remain unchanged. Numerical results also help to answer questions about optimal placement of sensor/actuator pairs for the active control of a flexible structure.

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I. SUMMARY

The combined system consisting of the baseline flexible structure modified by the system of active controllers is considered as a unified dynamical system. Techniques based on computer algebra (MACSYMA) are used to derive expressions for the transfer functions of the modified system, using the known transfer functions of the baseline flexible structure and the feedback gains of the active controller. The roots of the characteristic polynomial of this transfer function give the system resonant frequencies and damping parameters. Using the computer algebraic system MACSYMA, expressions for these parameters which are explicitly dependent on the output feedback gains of the active controller, are presented. For lightly coupled modes, simple relations are obtained between the modal parameters and the coordinates of the sensor/actuator pairs as well as the displacement and velocity feedback gains. These results permit the parametric study of the placement of the resonant frequencies and damping parameters of the combined system, as functions of the feedback gains. Numerical examples are used to illustrate the application of these results to the calculation of active controller feedback gains based on the requirement that certain modes have specified modal damping while the closed-loop frequencies remain unchanged. Also, using the example of a cantilevered uniform beam controlled by a single sensor/actuator pair, numerical results are used to illustrate the sensitivity of the closed-loop modal parameters to the placement of the sensor/actuator pair as well as the feedback gains. Such results help to answer questions about optimal placement of sensor/actuator pairs for the active control of a flexible structure. **[Key words: Active Control; Smart Structures; Computer Algebra; Dynamics of Flexible Structures]**

II. NOMENCLATURE

a_i, a_j	=	measurement coordinates for i^{th} and j^{th} controllers
b_i, b_j	=	force application coordinates for i^{th} and j^{th} controllers
$C_j^{(i)}$	=	symbolic coefficient of j^{th} power of p in the i-term derivation
$G(x,\xi,t,\tau)$	=	Green's function
g_i, g_j	=	displacement feedback gains of i^{th} and j^{th} controllers
h_i, h_j	=	velocity feedback gains of i^{th} and j^{th} controllers
$\mathbf{I}$	=	identity matrix
L	=	number of discrete attachments (controllers)
$L_{x,t}\{\}$	=	partial differential operator
p	=	Laplace variable
p_{0k}	=	kth parameter of baseline system
$Q(x,t)$, $Q(x,p)$	=	system response and its Laplace transform
t	=	time
$W(x,\xi,p)$	=	system transfer function
$W_{ci}(p)$	=	transfer function of i^{th} controller
$w(x,t)$, $w(x,p)$	=	forcing function and its Laplace transform
x, x_1, x_2	=	spatial coordinates
α_i	=	coefficient of the i^{th} power of p in characteristic polynomial
β, β_k	=	modal weights

$\delta(\,)$	=	Dirac delta function
$\{\gamma\}, \gamma_j$	=	vector, with elements defined in Equation(15)
η	=	spatial coordinate
φ_{0k}	=	k^{th} orthonormal modal function for baseline system
$\sigma, \sigma_{0n_1}, \sigma_{1n_1}$	=	exponential growth rate
τ	=	time
$\omega, \omega_{0n_1}, \omega_{1n_1}$	=	frequency
$\{\omega\}, \omega_i$	=	vector, with elements defined in Equation(13)
ξ	=	spatial coordinate
$[\,\Omega], \Omega_{i,j}$	=	matrix, with elements defined in Equation(12)

III. INTRODUCTION

This research effort has been concerned with the development of new techniques for the analysis of the dynamic response of distributed parameter flexible structures which are being controlled actively at discrete locations. This effort is a collaboration between Clark Atlanta University and AEDAR Corporation. This work grew out of the advancements made by AEDAR in the application of computer-algebraic techniques to the Green's function analysis of structural systems modified by discrete substructures attached to them (Fabunmi, 1989).

The technology of active control of structures has been the subject of considerable research interest over the past decade in part because of the need to suppress

excessive vibrations associated with the deployment of large flexible structures in space (Various Authors, 1986; Atluri and Amos (Ed.), 1988). Because space-borne structures cannot afford the weight penalties of classical vibration control devices such as absorbers or isolators, a lot of effort has been devoted to various means of actively controlling the dynamic characteristics of these structures. These techniques use an external source of energy to apply controlling forces (and/or moments) on the structure which are determined in some relationship to the measured or estimated response of the structure. More advanced implementations of active structural control involve the embedding of sensors, actuators and processors in the structure itself. Such "smart structures" are able to adjust the characteristics of their controllers e.g. feedback gains, in response to changing dynamical environments.

The most popular approach to the design of the active control schemes follows the paths of modern control theory which involves optimal state-space feedback control (Various Authors, 1984, 1986; Atluri and Amos (Ed.), 1988; O'Donoghue and Atluri, 1985; Horner and Walz, 1985), or output feedback control (Meirovitch, 1988; Garcia and Inman, 1990). A finite order mathematical model of the structure is required. The state variables are the [generalized] displacements and [generalized] velocities of the structure. In the case of state-space feedback, the state of the system is estimated from measurements at selected coordinates and this estimate is used to derive the feedback gains, using a method based on Pointryagin's principle for solving a constrained optimization problem. This method involves the computation of a positive-definite matrix satisfying the algebraic matrix-Riccatti equation (Junkins and Rew, 1988). Output feedback control does not use the entire state-space estimate for feedback; instead only the measured responses are used. The advantage is that the practical implementation of the controller is simpler and errors associated with the

estimation of unmeasured responses are eliminated (Garcia and Inman, 1990). For designs based on these methods, the control strategy is specified in terms of the minimization of an objective function. For a given objective function and a set of initial conditions, the controller feedback gains are calculated once and for all. A question that is often asked is whether or not optimal control necessarily implies intelligent control. For the control to be considered intelligent, the system must have the ability to alter its control strategies, and readjust the feedback gains, based on some reasoning. An example of such reasoning is to have a controlled structure which, upon sensing (using its embedded sensors) that the spectral distribution of its current external excitation is close to one or more of its resonant modes (the parameters of which has been stored in the memory of its embedded processor), can alter the strategy of its active controller such that the necessary feedback gains are computed (using its embedded processor) which will maximize the modal damping of the most highly excited modes. This type of approach requires that algorithms be available for directly calculating the controller feedback gains, based on specifications of required closed-loop modal parameters. The development of such algorithms has been one of the motivations of this research effort.

Recent advances in computer algebra have made available symbolic manipulation facilities which extend the tools of algebra and integro-differential calculus beyond the traditional limits (Rand, 1984; Pavelle, 1985). Techniques based on computer algebra were reported by Fabunmi (1989), which permit the derivation of the transfer functions (Laplace transform of the Green's Functions) of the system resulting from the attachment of discrete dynamic substructures to a distributed parameter base-line structure. It is assumed that the algebraic forms of the transfer function of the base-line structure as well as those of the discrete attachments are known. The mathematical

form chosen for system transfer functions permits the direct determination of the system parameters as the complex values of the Laplace variable at which singularities of the transfer function occur. For the class of controllers where the measured output are the feedback variables such as displacements and velocities, the resulting system is mathematically equivalent to that of the attachment of discrete "substructures", the transfer functions of which are given by expressions involving the gain constants and the Laplace variable.

This report has been organized as follows. In the section IV, the equations which were used to derive the effects of active controllers on the system parameters are derived. This section includes some pertinent material from (Fabunmi, 1989), for completeness. Section V presents the expressions for the characteristic polynomials of the closed-loop system, obtained using MACSYMA on the Symbolics 3620 workstation. This is followed by examples of how these results can be used to calculate the controller feedback gains based on specified parameters of the closed-loop system (section VI), and in section VII, using the example of a cantilevered uniform beam controlled by a single sensor/actuator pair, numerical results are used to illustrate the sensitivity of the closed-loop modal parameters to the placement of the sensor/actuator pair as well as the feedback gains. The basic conclusions of this report which are presented in the section VIII, are that (1) new tools based on computer algebra have been developed, for the analysis of system characteristics of actively controlled structures, (2) alternative techniques for the design of active controllers have been presented, which make it possible to design an adaptive controller for which different schedules of feedback gains can be used to adjust the system parameters as needed, in order to minimize dynamic response to external excitations, and (3) these results make it possible to select locations for sensors and actuators, at

which the closed-loop system parameters have the desired sensitivities to the velocity and displacement feedback gains.

IV. ANALYSIS OF ACTIVE CONTROLLER EFFECTS

The objective of this section is to present the formulation of the equations that were used to derive the transfer function of the system resulting from the attachment of a finite number of discrete linear output feedback controllers to a distributed parameter, baseline system such as a flexible structure. These derivations follow the same lines as those presented in (Fabunmi, 1989). The dynamic response of a distributed parameter system are solutions to partial integro-differential equations which can be represented operationally as:

$$\boldsymbol{L}_{x,t}\{Q(x_2,t)\} = w(x_1,t) \tag{1}$$

where $\boldsymbol{L}_{x,t}\{\ \}$ is an integro-differential operator which maps the responses $Q(x_2,t)$ on to the excitations $w(x_1,t)$ subject to appropriate boundary and initial conditions on $Q(x_2,t)$; x_1 is in the spatial domain of the excitations, x_2 is in the spatial domain of the responses and t is time. For linear operators, the Green's function $G(x,\xi,t,\tau)$ is defined such that:

$$\boldsymbol{L}_{x,t}\{G(x_2,\xi,t,\tau)\} = \delta(x_1-\xi)\delta(t-\tau) \tag{2}$$

where $\delta(\ )$ is the Dirac-delta function. The response of the system can be conveniently written in terms of the Green's function as:

$$Q(x_2,t) = \iint G(x_2,\xi,t,\tau)w(\xi,\tau)d\xi d\tau \tag{3}$$

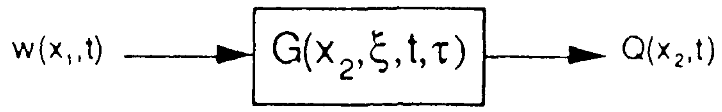
since,

$$
\begin{aligned}
\mathcal{L}_{x,t}\{Q(x_2,t)\} &= \iint \mathcal{L}_{x,t}\{G(x_2,\xi,t,\tau)\}w(\xi,\tau)d\xi d\tau \\
&= \iint \delta(x_1-\xi)\delta(t-\tau)w(\xi,\tau)d\xi d\tau \\
&= w(x_1,t)
\end{aligned}
\tag{4}
$$

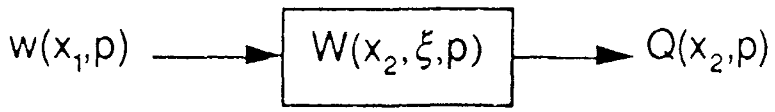
Butkovskyi (1983) has proposed the introduction of a linear distributed block - in analogy to the lumped parameter block in classical control theory - to represent the input-output relationship between $Q(x_2,t)$ and $w(x_1,t)$. Thus the schematic of Figure 1(a) is equivalent to the relationship expressed in Equation (3) . For dynamical systems whose responses to stationary excitations are stationary, i.e. the Green's function is stationary in time, the analysis can be simplified by considering the Laplace transform of the equations of motion. The role of the Green's function is now played by the Transfer function, and the relationship of the Laplace transform of the response $Q(x_2,p)$ to that of the excitation $w(x_1,p)$ is given by:

$$
Q(x_2,p) = \int G(x_2,\xi,p)w(\xi,p)d\xi \tag{5}
$$

where $p = \sigma + i\omega$ is the Laplace variable; σ is the exponential growth rate, and ω is the frequency. This relationship is also depicted schematically in Figure 1(b).



(a) Green's Function Representation



(b) Transfer Function Representation

Figure 1. Linear Distributed Block.

Modelling of Active Controller Attachments

The implementation of the active controller design involves the application of excitation forces at some spatial coordinate $x = b_i$, which are proportional to displacements and velocities measured at $x = a_i$. For example, the i^{th} controller excitation force could be written as:

$$w_c(b_i,t) = g_i Q(a_i,t) + h_i \dot{Q}(a_i,t) \tag{6}$$

where g_i and h_i are the displacement and velocity feedback gains of the i^{th} controller respectively, i = 1,2,...L, L being the total number of controllers. The Laplace transform of Equation (6) results in a relationship which is used to define the transfer function of the i^{th} controller as:

$$\begin{aligned} W_a(p) &= \frac{w_c(b_i,p)}{Q(a_i,p)} \\ &= g_i + h_i p \end{aligned} \tag{7}$$

The schematic which represents the combined interconnected system of the baseline structure and the L active controllers is shown in Figure2. The transfer function of the combined system shown in Figure 2 is given by the following integral equation (Butkovskyi, 1983):

$$W(x,\xi,p) = \int W_T(x,\eta,p) W(\eta,\xi,p) d\eta + W_0(x,\xi,p) \tag{8}$$

where,

$$\begin{aligned} W_T(x,\xi,p) &= \int W_0(x,\eta,p) \sum_{i=1}^{L} \delta(\eta - b_i) W_a(p) \delta(\xi - a_i) d\eta \\ &= \sum_{i=1}^{L} W_0(x,b_i,p) W_a(p) \delta(\xi - a_i) \end{aligned} \tag{9}$$

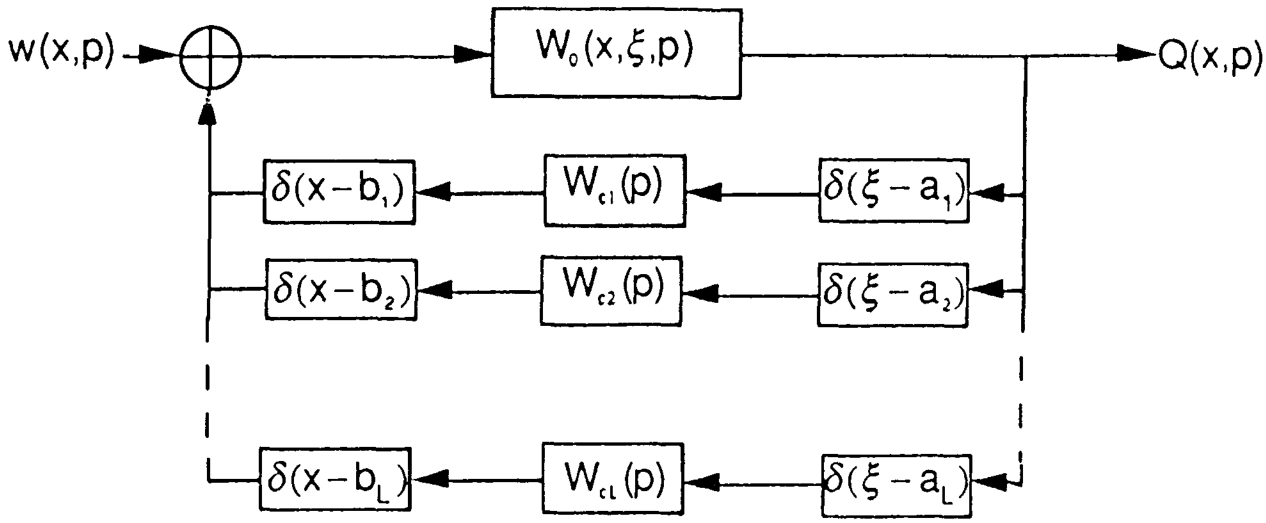


Figure 2. Schematic of Interconnection of Linear Feedback Controllers to Distributed Parameter Baseline Structure.

Substituting Equation(9) into Equation (8) and performing the integration, the result is,

$$W(x,\xi,p)=\sum_{i=1}^{L}W_0(x,b_i,p)W_{ci}(p)W(a_i,\xi,p)+W_0(x,\xi,p) \tag{10}$$

In order to solve for the quantities $W(a_i,\xi,p)$, i = 1,2,....L, both sides of Equation (10) are successively multiplied by $\delta(x - a_m)$ and integrations are performed over the x domain for m = 1,2,....L to get:

$$W(a_m,\xi,p)=\sum_{i=1}^{L}W_0(a_m,b_i,p)W_{ci}(p)W(a_i,\xi,p)+W_0(a_m,\xi,p) \tag{11}$$

Equation(11) is a system of L linear equations defining L unknown quantities. If an (LxL) matrix $[\Omega]$ is defined such that its elements are,

$$\Omega_{i,j} = W_0(a_j, b_i, p) W_{ci}(p) \tag{12}$$

and if an (Lx1) vector $\{\omega\}$ is defined such that its elements are,

$$\omega_i = W(a_i, \xi, p) \tag{13}$$

then the system of Equation (11) for m = 1,2,....L can be written in a compact form as:

$$\{\omega\} = [\Omega]\{\omega\} + \{\gamma\} \tag{14}$$

where $\{\gamma\}$ is an (Lx1) vector whose elements are,

$$\gamma_j = W_0(a_j, \xi, p) \tag{15}$$

From (14),

$$\{\omega\} = [\mathbf{I} - \Omega]^{-1}\{\gamma\} \tag{16}$$

where $\mathbf{I}$ is the (LxL) identity matrix.

Application of Computer Algebra

The general algebraic form of the transfer function of the baseline distributed system is taken to be (Chen, 1966; Stakgold, 1979; Butkovskyi, 1982, 1983; Keener, 1988; Fabunmi, 1989):

$$W_0(x, \xi, p) = \sum_{k=1}^{\infty} \left(\frac{1}{\beta_k^2} \frac{\varphi_{0k}(x)\varphi_{0k}(\xi)}{p^2 - p_{0k}^2} \right) \tag{17}$$

where $\varphi_{0k}(x)$ is the k^{th} orthonormal modal function for the baseline structure, p_{0k} is the

corresponding modal parameter and $\beta_K{}^2$ is the weighting factor or generalized mass. Although the summation in Equation (17) includes an infinite number of terms, the practical implementation of that expression requires that only a finite number of terms be retained. The ability to retain a given number of modes in the algebraic derivation depends on the power and memory of the computer as well as the number of discrete modifications to the baseline structure. The substitution of Equation (17) into Equation (16) and the subsequent evaluation and simplification of the transfer function of the combined system as shown in Equation (10) is performed using the following set of MACSYMA routines:

```
W0(EXX,XXSI,PEE):=BLOCK(
    PURPOSE:"EXPRESSION FOR TRANSFER FUNCTION FOR BASELINE
    STRUCTURE - I.E. THE FUNCTION W0(X,XSI,P)"
    RAT(SUM('PHI(EXX,K)*'PHI(XXSI,K)/(PEE^2-('P0[K]^2),K,N1,N2))/'BSQ)$

GAMMA_VECTOR(XXXSI,ARGP):=BLOCK(
    GAMMA:ZEROMATRIX(NS,1),
    FOR J THRU NS
    DO SETELMX(W0('A[J],XXSI,ARGP),J,1,GAMMA))$

OMEGA_MATRIX(ARGP):=BLOCK(
    CAP_OMEGA:ZEROMATRIX(NS,NS),
    FOR I THRU NS
    DO (
    FOR J THRU NS
    DO (W0IJ:W0('A[J],'B[I],ARGP),
    SETELMX(W0IJ*WCP[I],I,J,CAP_OMEGA))))$
```

```
W(EXX,XXSI,PEE,N11,N22,N33):=BLOCK(
        SCALARMATRIXP:FALSE,N1:N11,N2:N22,NS:N33,
        FOR KK FROM N1 THRU N2
        DO STARTP(KK),
        FOR N THRU NS
        DO WCP[N]:RAT(SUBST(PEE,P,WC[N])),
        OMEGA_ MATRIX(PEE), GAMMA_VECTOR(XXSI,PEE),
        MATRIX:IDENT(NS)-CAP_OMEGA,OMEGA:ZEROMATRIX(NS,1),
        INVERSE_MATRIX:RAT(ADJOINT(MATRIX))/RAT(DETERMINANT(MATRIX)),
        POLY:DENOM(INVERSE_MATRIX[1,1]),
        OMEGA:RAT(INVERSE_MATRIX . GAMMA),
        W1:RAT(SUM(W0(EXX,B[I],PEE)*WCP[I]*OMEGA[I,1],I,1,NS)+
        W0(EXX,XXSI,PEE)), "DONE")$
```

W1 gives the expression for the transfer function of the combined system; POLY is the characteristic polynomial of the combined system. In order to cast the resultant transfer function into the form of Equation(17) for the combined system, the system parameters p_{1k} are determined as the roots of the characteristic polynomial of the system. In the above routines, it is possible to consider any range of terms [N11,N22] in the baseline transfer function series, as well as any number [N33] of discrete attachments to the baseline system. The computer-algebraic results that will be presented in the next section have been generalized to the case of an arbitrary number of discrete attachments. This is done by mathematical induction, based on the results provided by MACSYMA for different number of attachments specified in the function calls.

V. COMPUTER-ALGEBRAIC RESULTS

Some results of the derivation of the characteristic polynomials for an arbitrary number of attachments using one- and two terms in the base-line transfer function series, are presented in this section. These derivations were performed on the Symbolics 3620. A uniform one-dimensional baseline structure was assumed in this study, so that $\beta_k{}^2=\beta^2$ for all the k's.

One-term derivation:

$$\text{POLYNOMIAL}(1) = C_2^{(1)} p^2 + C_0^{(1)}(p) \tag{18}$$

where,

$$C_0^{(1)}(p) = -\beta^2 p_{0n_i}^2 - \sum_{i=1}^{L} \{W_{ci}(p)\varphi_{0n_i}(a_i)\varphi_{0n_i}(b_i)\} \tag{19}$$

and,

$$C_2^{(1)} = \beta^2 \tag{20}$$

Note that the polynomial in Equation (18) is not yet fully explicit in p until the expressions for $W_{ci}(p)$ are substituted into Equation (19). If these functions are as shown in Equation(7) then,

$$C_0^{(1)}(p) = -\beta^2 p_{0n_i}^2 - \sum_{i=1}^{L} \{(g_i + h_i p)\varphi_{0n_i}(a_i)\varphi_{0n_i}(b_i)\} \tag{21}$$

So that for this class of active controller design, and using the one-term approximation to the baseline transfer function, the characteristic polynomial in p whose roots are the

system parameters is given by:

$$\text{POLYNOMIAL}(1) = \left\{\beta^2\right\} p^2 - \left\{\sum_{i=1}^{L}\{h_i\, \varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\}\right\} p - \left\{\beta^2 p_{0n_1}^2 + \sum_{i=1}^{L}\{g_i\, \varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\}\right\} \tag{22}$$

Two-term derivation:

$$\text{POLYNOMIAL}(2) = C_4^{(2)} p^4 + C_2^{(2)}(p)\, p^2 + C_0^{(2)}(p) \tag{23}$$

where,

$$C_4^{(2)} = \left(\beta^2\right)^2 \tag{24}$$

$$\begin{aligned} C_2^{(2)}(p) &= -\left(\beta^2\right)^2 \sum_{j=1}^{2}\left\{p_{0n_j}^2\right\} - \sum_{i=1}^{L}\left\{W_{ci}(p)\sum_{j=1}^{2}\left[\beta^2\varphi_{0n_j}(a_i)\varphi_{0n_j}(b_i)\right]\right\} \\ &= -\left(\beta^2\right)^2 \sum_{j=1}^{2}\left\{p_{0n_j}^2\right\} - \sum_{i=1}^{L}\left\{W_{ci}(p)\left\lfloor \beta\varphi_{0n_1}(a_i) \quad \beta\varphi_{0n_2}(a_i) \right\rfloor \begin{Bmatrix}\beta\varphi_{0n_1}(b_i) \\ \beta\varphi_{0n_2}(b_i)\end{Bmatrix}\right\} \end{aligned} \tag{25}$$

and,

$$C_0^{(2)}(p) = \left(\beta^2 p_{0n_1}^2\right)\cdot\left(\beta^2 p_{0n_2}^2\right)$$
$$+\sum_{i=1}^{L}\left\{W_{ci}(p)\left\lfloor \beta\varphi_{0n_1}(a_i) \quad \beta\varphi_{0n_2}(a_i)\right\rfloor\begin{bmatrix} p_{0n_2}^2 & 0 \\ 0 & p_{0n_1}^2\end{bmatrix}\begin{Bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{Bmatrix}\right\}$$
$$+\sum_{i=1}^{L}\sum_{j>i}^{L}\left\{\begin{matrix} W_{ci}(p)W_{cj}(p)\left(\left\lfloor \varphi_{0n_1}(a_i) \quad -\varphi_{0n_2}(a_i)\right\rfloor\begin{Bmatrix}\varphi_{0n_2}(a_j)\\ \varphi_{0n_1}(a_j)\end{Bmatrix}\right) \\ \times\left(\left\lfloor \varphi_{0n_1}(b_i) \quad -\varphi_{0n_2}(b_i)\right\rfloor\begin{Bmatrix}\varphi_{0n_2}(b_j)\\ \varphi_{0n_1}(b_j)\end{Bmatrix}\right)\end{matrix}\right\} \tag{26}$$

As mentioned earlier, the explicit dependence of the polynomial on the Laplace variable p will be determined when the appropriate expressions are substituted for the functions $W_{ci}(p)$. If the controller transfer functions given in Equation(7) are substituted into Equation (25) and (26), then following a collection of the coefficients of the powers of p in the polynomial of Equation (23), the result is:

$$\text{POLYNOMIAL}(2) = \alpha_4 p^4 + \alpha_3\ p^3 + \alpha_2\ p^2 + \alpha_1\ p + \alpha_0 \tag{27}$$

where,

$$\alpha_4 = \left(\beta^2\right)^2 \tag{28}$$

$$\alpha_3 = -\sum_{i=1}^{L}\left\{h_i\left\lfloor \beta\varphi_{0n_1}(a_i) \quad \beta\varphi_{0n_2}(a_i)\right\rfloor\begin{Bmatrix}\beta\varphi_{0n_1}(b_i)\\ \beta\varphi_{0n_2}(b_i)\end{Bmatrix}\right\} \tag{29}$$

$$\alpha_2 = -(\beta^2)^2 \sum_{j=1}^{2} \left\{ p_{0n_j}^2 \right\} - \sum_{i=1}^{L} \left\{ g_i \left\lfloor \beta \varphi_{0n_1}(a_i) \quad \beta \varphi_{0n_2}(a_i) \right\rfloor \begin{Bmatrix} \beta \varphi_{0n_1}(b_i) \\ \beta \varphi_{0n_2}(b_i) \end{Bmatrix} \right\}$$

$$+ \sum_{i=1}^{L} \sum_{j>i}^{L} \left\{ \begin{array}{l} h_i h_j \left(\left\lfloor \varphi_{0n_1}(a_i) \quad -\varphi_{0n_2}(a_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(a_j) \\ \varphi_{0n_1}(a_j) \end{Bmatrix} \right) \\ \qquad \times \left(\left\lfloor \varphi_{0n_1}(b_i) \quad -\varphi_{0n_2}(b_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(b_j) \\ \varphi_{0n_1}(b_j) \end{Bmatrix} \right) \end{array} \right\} \tag{30}$$

$$\alpha_1 = \sum_{i=1}^{L} \left\{ h_i \left\lfloor \beta \varphi_{0n_1}(a_i) \quad \beta \varphi_{0n_2}(a_i) \right\rfloor \begin{bmatrix} p_{0n_2}^2 & 0 \\ 0 & p_{0n_1}^2 \end{bmatrix} \begin{Bmatrix} \beta \varphi_{0n_1}(b_i) \\ \beta \varphi_{0n_2}(b_i) \end{Bmatrix} \right\}$$

$$+ \sum_{i=1}^{L} \sum_{j>i}^{L} \left\{ \begin{array}{l} (g_i h_j + g_j h_i) \times \left(\left\lfloor \varphi_{0n_1}(a_i) \quad -\varphi_{0n_2}(a_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(a_j) \\ \varphi_{0n_1}(a_j) \end{Bmatrix} \right) \\ \qquad \times \left(\left\lfloor \varphi_{0n_1}(b_i) \quad -\varphi_{0n_2}(b_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(b_j) \\ \varphi_{0n_1}(b_j) \end{Bmatrix} \right) \end{array} \right\} \tag{31}$$

and,

$$\alpha_0 = \left(\beta^2 p_{0n_1}^2\right) \cdot \left(\beta^2 p_{0n_2}^2\right)$$

$$+ \sum_{i=1}^{L} \left\{ g_i \left\lfloor \beta \varphi_{0n_1}(a_i) \quad \beta \varphi_{0n_2}(a_i) \right\rfloor \begin{bmatrix} p_{0n_2}^2 & 0 \\ 0 & p_{0n_1}^2 \end{bmatrix} \begin{Bmatrix} \beta \varphi_{0n_1}(b_i) \\ \beta \varphi_{0n_2}(b_i) \end{Bmatrix} \right\}$$

$$+ \sum_{i=1}^{L} \sum_{j>i}^{L} \left\{ \begin{array}{l} (g_i g_j) \times \left(\left\lfloor \varphi_{0n_1}(a_i) \quad -\varphi_{0n_2}(a_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(a_j) \\ \varphi_{0n_1}(a_j) \end{Bmatrix} \right) \\ \qquad \times \left(\left\lfloor \varphi_{0n_1}(b_i) \quad -\varphi_{0n_2}(b_i) \right\rfloor \begin{Bmatrix} \varphi_{0n_2}(b_j) \\ \varphi_{0n_1}(b_j) \end{Bmatrix} \right) \end{array} \right\} \tag{32}$$

For the class of active controllers considered in this study, i.e those with displacement and velocity feedback, the preceding expressions permit the explicit understanding of how the feedback gains affect the resulting system parameters. As a matter of fact, since polynomials up to the fourth order can be solved in closed form by computer

algebra, it follows that closed form expressions can be obtained for the combined system parameters in terms of the feedback gains, as well as the location of the measurement and actuation points. In the following section, the results for the one-term derivation will be used to illustrate a possible method for determining the controller feedback gains when it is desired to achieve specified closed-loop system parameters. As will be seen, this method involves relatively simple calculations which can easily be programmed into embedded processors of "smart" structures.

VI. CALCULATION OF CONTROLLER FEEDBACK GAINS

Consider a baseline structure with negligible damping, i.e. $p_{0n1} = i\omega_{0n1}$. By virtue of the weak coupling of the modes of an undamped structure, the one-term derivation is adequate for estimating the closed-loop system parameters. Equating the polynomial of Eq.22 to zero and solving for the real and imaginary parts of p_{1n1}, the frequencies and exponential growth rates of the closed-loop modes are;

$$\omega_{1n_1} = \sqrt{\omega_{0n_1}^2 - \left\{\frac{1}{2\beta^2}\sum_{i=1}^{L}\{h_i\,\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\}\right\}^2 - \frac{1}{\beta^2}\sum_{i=1}^{L}\{g_i\,\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\}} \tag{33}$$

$$\sigma_{1n_1} = \frac{1}{2\beta^2}\sum_{i=1}^{L}\{h_i\,\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i)\} \tag{34}$$

The damping of the closed-loop system is controlled by the velocity feed-back gains, whereas the frequency is affected by both the velocity and displacement feedback gains. If L velocity feedback gains are to be calculated directly based on specified values of the growth rates of L system modes, Equation (34) provides a set of L

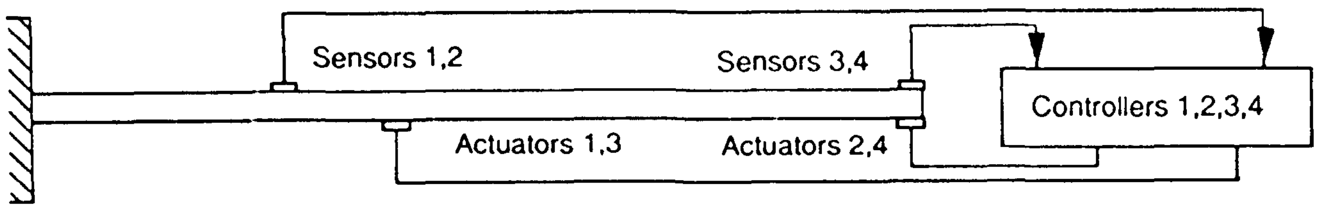
equations for the L desired values of h_i. It may also be desired that there be no shift in the frequencies of these or some other L modes. In that case, a set of L equations for the L values of the displacement feedback gains, g_i, can be set up as follows:

$$-\left\{\frac{1}{2\beta^2}\sum_{i=1}^{L}\{h_i\,\varphi_{0n_i}(a_i)\varphi_{0n_i}(b_i)\}\right\}^2-\frac{1}{\beta^2}\sum_{i=1}^{L}\{g_i\,\varphi_{0n_i}(a_i)\varphi_{0n_i}(b_i)\}=0 \tag{35}$$

Having obtained the values of the velocity and displacement feedback gains in this manner, it is necessary to check the exponential growth rates of the modes that were not included in the analysis, using Equation (34). The purpose of this check is to verify that there are no modes for which the exponential growth rate is positive - an indication that instability of that mode can be induced by the controller.

Example

As an illustration of this approach, consider the cantilevered uniform beam shown in Figure 3. The beam is 100 meters long, with the following cross sectional properties: flexural rigidity, EI = 1.0e8 N-m^2 ; mass per unit length, ρA = 1 kg/m. The natural frequencies and the orthonormal modes of the first ten modes of this beam are shown in Table 1. The active control system consists of two sensors and two actuators. The sensors are located at spanwise coordinates 30 and 100 m. The actuators are located at spanwise coordinates 40 and 100m. Consistent with the notation in this paper, the active control system is made up of four (4) controllers: controller #1 generates a force signal at x=40 based on the sensor signal at x=30; controller #2 generates a force signal at x=100 based on the sensor signal at x=30; controller #3 generates a force signal at x=40 based on the sensor signal at x=100; and controller #4 generates a force signal at x=100 based on the sensor signal at x=100.



Index of Controller i	Sensor Location a_i	Actuator Location b_i
1	30	40
2	30	100
3	100	40
4	100	100

Figure 3. Example of cantilevered uniform beam with two actuators and two sensors.

Table 1. The First Ten Orthonormal Modes of the example Uniform Cantilevered Euler-Bernoulli Beam

Flexural Rigidity EI=1.0e8 N-m^2; Mass per unit length, ρA=1 kg/m; Length, L = 100 m

n_1	ω_{0n_1}	A	B	C	D	$\varphi_{0n_1}(X)$
1	3.516	0.02417	4.138	0.01875	3.038	$A(B(\cos(Cx)-\cosh(Cx))-D(\sin(Cx)-\sinh(Cx)))$
2	22.034	0.00186	53.645	0.04694	54.636	
3	61.697	7.67e-5	1289.986	0.07854	1288.985	
4	120.902	0.10235	-	0.10996	-	$A(\cos(Cx)-\sin(Cx)-e^{-Cx})$
5	199.860	0.10182	-	0.14137	-	
6	298.556	C.10148	-	0.17279	-	
7	416.991	0.10125	-	0.20420	-	
8	555.165	0.10108	-	0.23562	-	
9	713.079	0.10095	-	0.26704	-	
10	890.732	0.10085	-	0.29845	-	

Because of the linearity of both the structural model and the controller design, there need only be one physical actuator at x=40 and another one at x=100. The force signals for actuators 1 and 3 are summed and applied to the physical actuator at x=40; similarly, the force signals for actuators 2 and 4 are summed up and applied to the physical actuator at x=100. Also, only one physical sensor at x=30 is needed for controllers 1 and 2, and one physical sensor at x=100 is needed for controllers 3 and 4. The table in Figure 3 specifies the sensor and actuator positions for the four controllers in this example. The objective of this example is to show how to calculate the velocity and displacement feed-back gains h_i, g_i, i=1,2,3,4 such that: (1) the modal damping of four selected modes are as specified in advance; (2) the natural frequencies of four selected modes (not necessarily the same ones as in (1)) are unchanged; and (3) none of the first ten modes is destabilized by the active control system. The restriction to the specification of the parameters of only four modes is due to the fact that there are only four independent controllers under consideration. The feedback gains calculated in this manner are not optimal in the usual sense of minimizing some objective function which is related to both the response and the control power. The merit of this approach lies in the ability to concentrate available control power on the damping of certain modes which are considered most responsive to a given external excitation, without destablizing the other modes. Thus if the nature of external excitation were to change, and hence require that some other modes be critically damped, this method affords a means of adjusting the feed-back gains appropriately.

For each of the four modes for which a desired damping ratio is specified in advance, Equation 34 gives four linear equations for the four unknown coefficients h_i, i=1,2,3,4;

which are the velocity feed-back gains. After the velocity feed-back gains have been determined, Equation 35 is then used to obtain the additional relations needed to determine the displacement feed-back gains g_i, $i=1,2,3,4$; based on the conservation of the natural frequencies of the desired modes. For this example, it is required that the first mode be critically damped, and that the next three modes be moderately damped. It is also required that there be no shift in the frequencies of the first four modes of the beam. Table 2(a) shows the specified damping ratios for the first four modes of the cantilevered beam. For the first mode to be critically damped, the damping ratio is specified to be unity. The results of the calculations of the velocity and displacement feed-back gains are given in Table 2(b) for this example. Having obtained the values of the four pairs of displacement and velocity feedback gains for the four controllers, it is now possible to use Equations 33 and 34 to recalculate the closed loop frequencies and exponential growth rates of all the other modes. The results of this calculation for the first ten modes of the example uniform cantilever beam, are presented in Table 2(c). This example shows that the feedback gains calculated will yield the desired damping ratios and frequency shifts without destabilizing any of the first ten modes of the example beam. So far, there is no guarantee that all the higher order modes will be stable. If upon checking further, some higher order mode is found with a negative damping ratio, it is prudent to perform the calculation again, including the unstable mode among the modes for which the damping ratio is specified. Further research is needed to develop a more systematic approach for ensuring the stability of the modes for which specific damping values were not specified in advance.

Table 2 (a). Target Damping Ratios for Selected Modes

Mode No. n	Target Damping Ratio $-\frac{\sigma_n}{\omega_n}$
1	1.00
2	0.10
3	0.03
4	0.01

Table 2 (b). Velocity and Displacement Feed-back Gains

Controller No. i	Velocity Feed-back Gains - N-sec/m h_i	Displacement Feed-back Gains (to conserve frequencies of first 4 modes) - N/m g_i
1	279.87	1071.83
2	-73.60	-554.02
3	82.10	581.60
4	-193.41	-400.77

Table 2 (c). Closed Loop Parameters of First Ten Modes of the Example Beam

Mode No. n	Damping Ratio $-\frac{\sigma_n}{\omega_n}$	Percentage shift in natural frequency $\frac{(\Delta\omega_n)}{\omega_n}\times 100\%$
1	1.00	0.0
2	0.10	0.0
3	0.03	0.0
4	0.01	0.0
5	0.0029	0.0041
6	0.0033	0.0035
7	0.0050	4.39e-4
8	0.0025	0.0013
9	0.0038	0.0015
10	0.0029	1.37e-4

VII. SENSITIVITY ANALYSIS

The expressions given in Equations 33 and 34 show that the sensor/actuator placement controls how the feedback gains influence the frequencies and damping of the participating modes of the baseline structure. This effect is more clearly displayed by the sensitivity of the growth rate to the velocity feedback gains as well as the sensitivity of the frequencies to the displacement feedback gains:

$$\frac{\partial \sigma_{1n_i}}{\partial h_i} = -\frac{1}{2}\frac{\partial \omega^2_{1n_i}}{\partial g_i} = \frac{1}{2\beta^2}\varphi_{0n_i}(a_i)\varphi_{0n_i}(b_i) \tag{35}$$

When the sensor and actuator of the i-th controller are collocated (i.e. $a_i = b_i$), the right hand side (RHS) of Eq. 35 is positive, and all structural modes with non-zero values of the orthornormal modes at the controller location, are guaranteed to be damped by a negative velocity feedback gain. For controllers with non-collocated sensor/actuator pairs, the RHS of Eq.35 could be positive or negative for different modes of the baseline structure. The baseline modes for which the product $\varphi(a_i)\varphi(b_i)$ is positive, will be damped for negative velocity feedback gains, whereas those modes with a negative product of $\varphi(a_i)\varphi(b_i)$ could experience destabilization as a result of negative velocity feedback gain.

$$\frac{\partial \sigma_{1n_1}}{\partial g_i} = 0 \tag{36}$$

$$\frac{\partial \omega_{1n_1}^2}{\partial h_i} = -\left\{\frac{1}{2\beta^4}\sum_{k=1}^{L}\left\{h_k \varphi_{0n_1}(a_k)\varphi_{0n_1}(b_k)\right\}\right\}\varphi_{0n_1}(a_i)\varphi_{0n_1}(b_i) \tag{37}$$

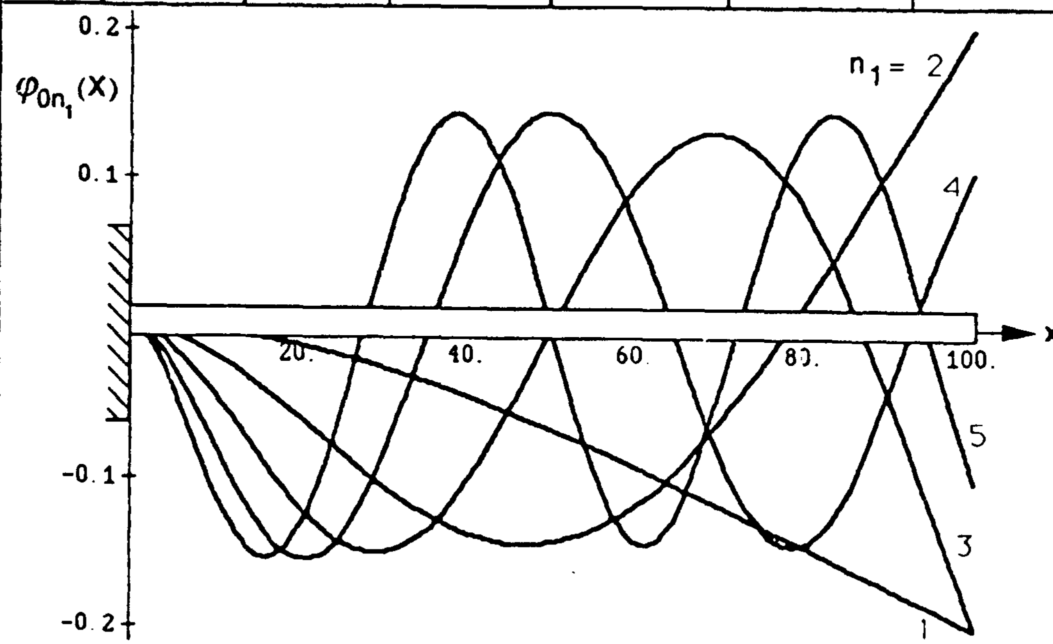
As expected, the growth rate is not affected by the displacement feedback. However, the closed loop frequency is sensitive to the velocity feedback as well as the displacement feedback. For a single sensor/actuator pair (L=1), the sign of the RHS of Eq.37 is dictated by the sign of the velocity feedback. In the following section, the numerical example of a uniform cantilevered Euler-Bernoulli beam is used to illustrate the feedback gain sensitivities of the closed-loop growth rate, as the sensor/actuator pair placement is varied, both for collocated and non-collocated controllers.

Example

Consider a uniform cantilevered Euler-Bernoulli beam, with parameters as shown in Fig. 4, where the first five orthonormal modes have also been displayed. Suppose a single sensor/actuator pair is to be employed to control the vibrations of this beam using displacement and velocity feedback. Three situations will be examined:

(1) the sensor and actuator are collocated at the same spanwise coordinate; (2) the sensor and actuator are not collocated, with the sensor always at the tip of the beam; and (3) the sensor and actuator are not collocated, with the sensor at midspan. In each of these situations, the growth rate sensitivity to velocity feedback gain for different locations of the actuator, will be studied. The growth rate sensitivity to velocity feedback gain is characterized by the partial derivative of the growth rate with respect to the velocity feedback gain. As given by Eq. 35, this quantity depends on the locations of the sensor and actuator. For vibration suppression, it is required that the

n_1	ω_{0n_1}	A	B	C	D	$\varphi_{0n_1}(X)$
1	3.516	0.02417	4.138	0.01875	3.038	$A(B(\cos(Cx)-\cosh(Cx))-D(\sin(Cx)-\sinh(Cx)))$
2	22.034	0.00186	53.645	0.04694	54.636	$A(B(\cos(Cx)-\cosh(Cx))-D(\sin(Cx)-\sinh(Cx)))$
3	61.697	7.67e-5	1289.986	0.07854	1288.985	$A(B(\cos(Cx)-\cosh(Cx))-D(\sin(Cx)-\sinh(Cx)))$
4	120.902	0.10235	-	0.10996	-	$A(\cos(Cx)-\sin(Cx)-e^{Cx})$
5	199.860	0.10182	-	0.14137	-	$A(\cos(Cx)-\sin(Cx)-e^{Cx})$



Flexural Rigidity, $EI = 10^8$

Mass per unit length, $\rho A = \beta^2 = 1$

Length, $L = 100$

Figure 4. The First Five Orthonormal Modes of a Uniform Cantilevered Euler-Bernoulli Beam

growth rate be negative, and also that its sensitivity to velocity feedback gain be of the same sign *for all the modes of the structure.* If the sign of this sensitivity is not the same for all modes, velocity feedback gains which suppress the vibration of certain modes, will tend to cause increased response others.

The plots of the growth rate sensitivity to velocity feedback gains for different locations of the actuator are shown in Figures 5-7. Figure 5 is for the case where the sensor and actuator pair are collocated, whereas in Figures 6 and 7, the sensor and actuator are not collocated. In Figure 6, the sensor is always at the tip, and Figure 7 is for the case where the sensor is always at midspan. When the sensor is collocated with the actuator, as in Figure 5 for all locations of the actuator, or in Figure 6 when the actuator is at the tip, or in Figure 7 when the actuator is at midspan, the growth rate sensitivity to velocity feedback retains the same sign for all the modes of the beam. Collocated sensor/actuator pairs near the tip of the cantilevered beam have higher sensitivities of the growth rate of the lower modes to velocity feedback, than the higher modes. This situation is reversed near the root of the beam. When the sensor is not collocated with the actuator, there are sign reversals of the growth rate sensitivity to velocity feedback, for different modes of the beam. In these cases, the same velocity feedback gain which increases the damping of certain modes, will reduce the damping of others. These effects are well known, and are not surprising. What is interesting in the present approach is that these sensitivities can be quantified explicitly. Trade-offs can be made with regard to the selection of modes which need maximum sensitivity, at the expense of other modes which may not be critical in a given application.

This example has been presented as a simple illustration of how the results of this analysis could be used in guiding the selection of sensor/actuator placements. In most

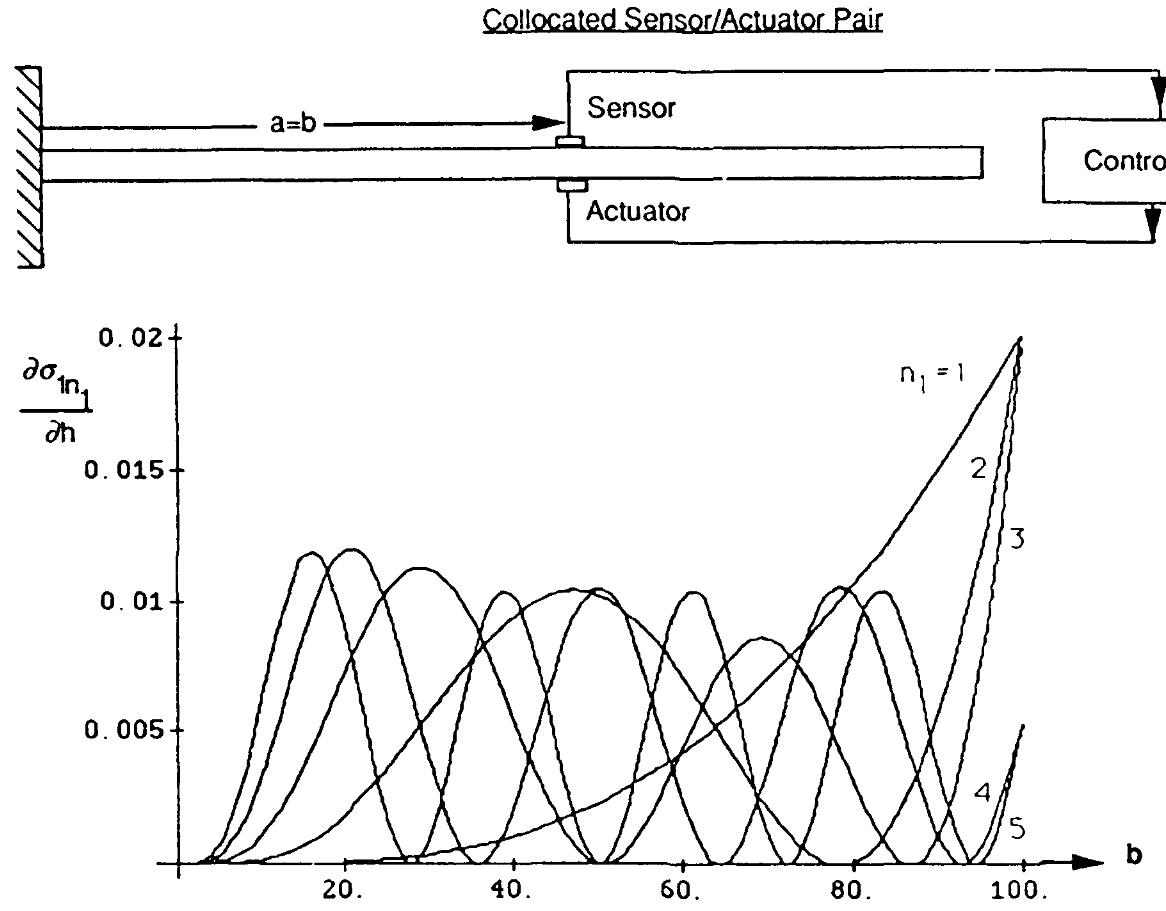


Figure 5. Effects of Actuator Placement on Closed-loop Damping Sensitivity to Velocity Feedback - Collocated Sensor/Actuator Pair

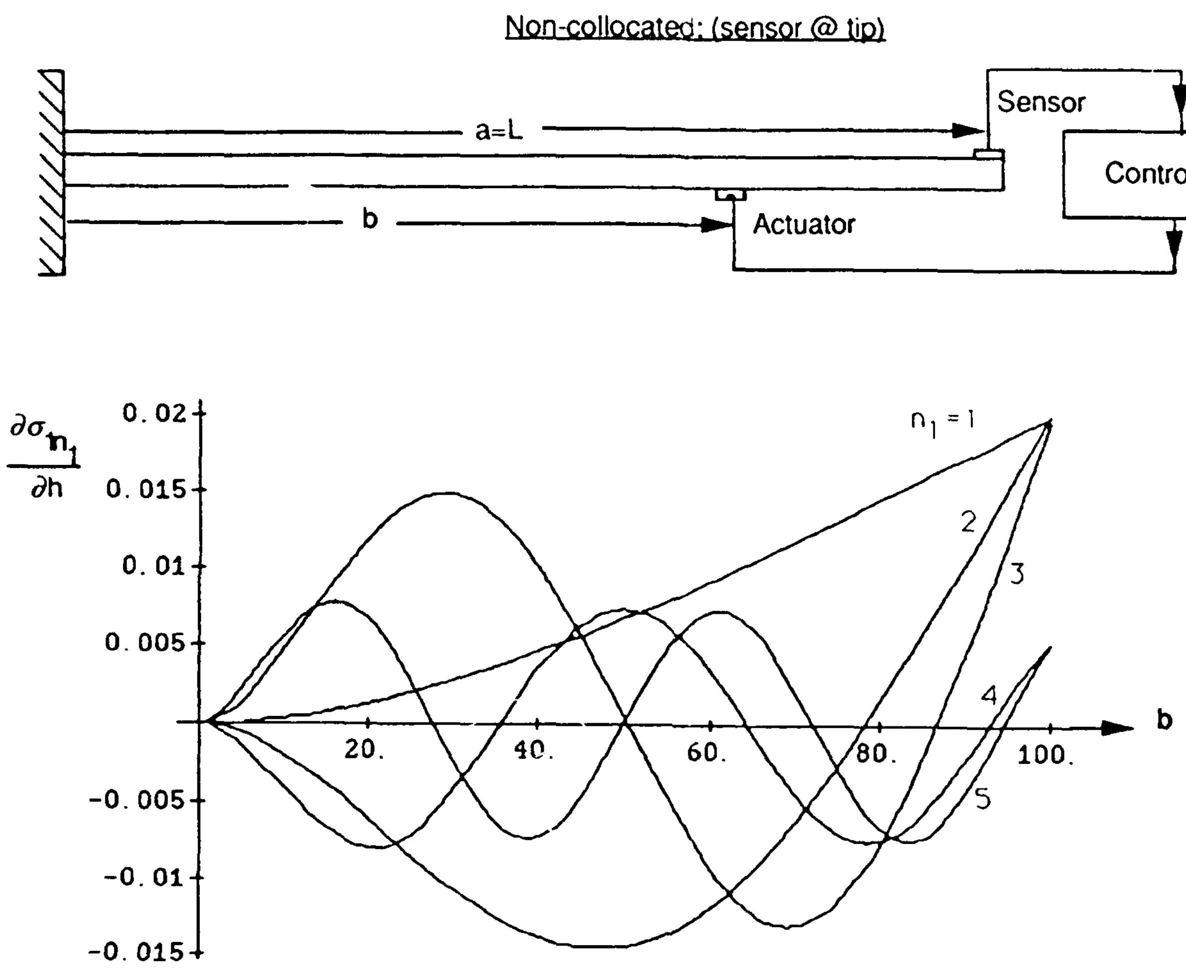


Figure 6. Effects of Actuator Placement on Closed-loop Damping Sensitivity to Velocity Feedback - Non Collocated Sensor/Actuator Pair; Sensor @ Tip

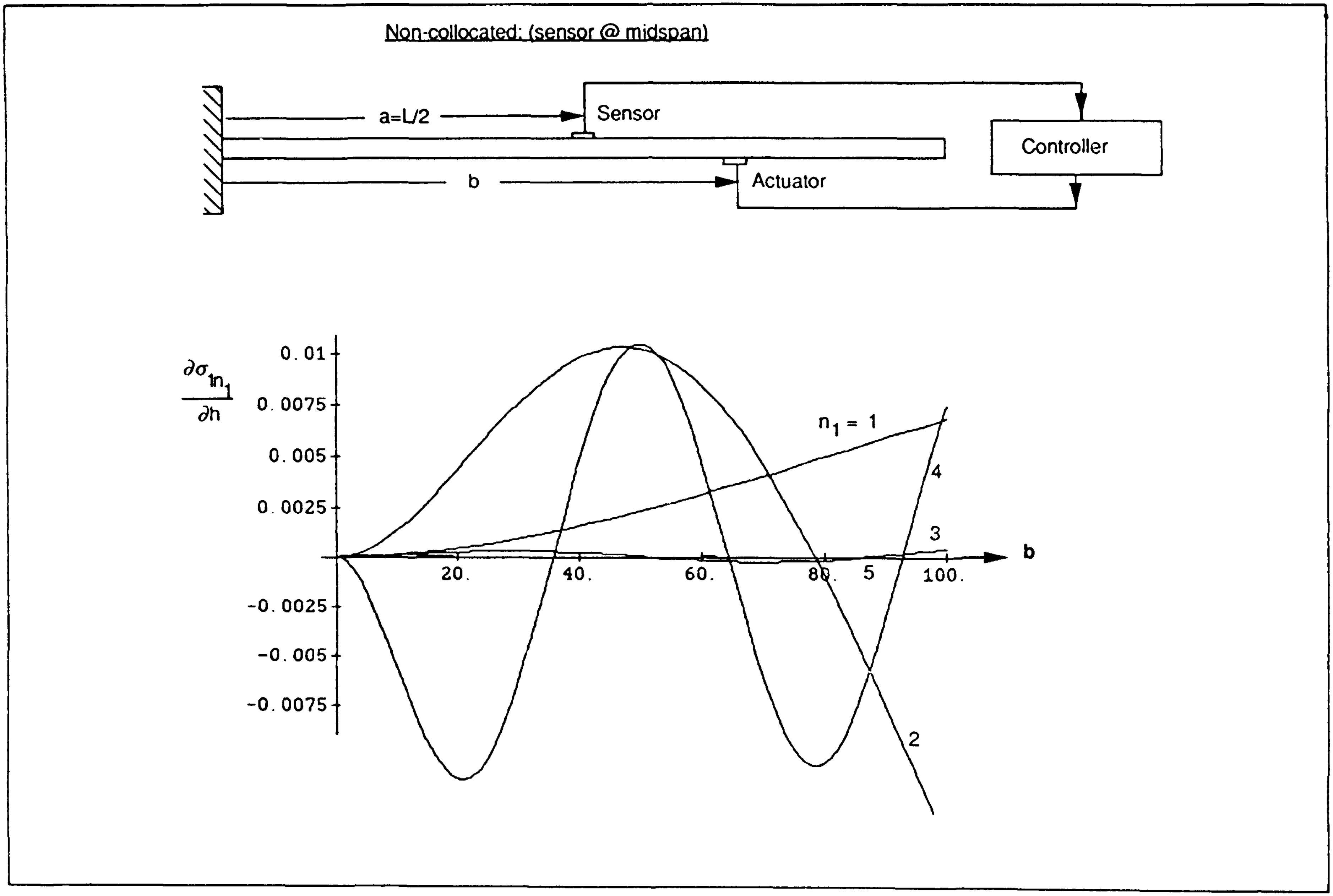


Figure 7. Effects of Actuator Placement on Closed-loop Damping Sensitivity to Velocity Feedback - Non Collocated Sensor/Actuator Pair; Sensor @ Midspan.

practical applications, the orthonormal modes may not be available in analytical form. It may become necessary to conduct experimental modal testing, in order to obtain the modal data required by this approach. Regardless of how the modal data is obtained, the simple expressions derived in this paper can be used to obtain preliminary assessments of the most advantageous placement of sensor/actuator pairs for active vibration control.

VIII. CONCLUSIONS

This report has presented results of computer-algebraic derivations of the characteristic parameters of systems consisting of a distributed baseline structure and output feedback linear active controllers. Expressions which show the explicit dependence of the system parameters on the displacement and velocity feed-back gains as well as the measurement and actuator coordinates were obtained for the cases when the transfer function of the baseline system has been approximated by retaining one- and two- terms in the infinite series which determine these transfer functions. An immediate application of these results is the calculation of displacement and velocity feed-back gains based on requirements that certain closed loop modes have specified damping ratios. Because of the simplicity of the calculations involved in this process, it becomes practical to conceive embedded systems which permit the "smart" structure to readjust its feedback gains in order to increase the damping of those of its modes which are most strongly excited by the external dynamical forces. The example of a cantilevered uniform beam was used to illustrate how to implement this methodology for a finite number of controllers. Further research is indicated to develop a systematic way of assuring that the active control system does not cause a

destablilzation of those modes for which the damping ratios were not specified in advance.

The sensitivities of closed-loop system parameters to displacement and velocity feedback gains are important considerations for the placement of sensors and actuators for active suppression of structural vibrations. Whereas optimal controller design methods using modern control theory can yield the optimal feedback gains, they do not provide insights into to the placement of the sensors and the actuators for the best effect. Using computer algebra, explicit expressions have been derived for the sensitivities of the closed-loop system parameters such as resonant frequency and exponential growth rates, to velocity and displacement feedback gains. These expressions make it possible to select locations for sensors and actuators, at which the closed-loop system parameters have the desired sensitivities to the velocity and displacement feedback gains. Numerical examples based on a cantilevered uniform beam were used to illustrate the application of this method, and to show that it is possible to conduct a rational assessment of the most advantageous locations of sensors and actuators for active control.

IX. ACKNOWLEDGEMENTS

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